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A sequential method to estimate the strength of the heat source based on symbolic computation

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Abstract—A sequential method is proposed to determine the strength of the heat source in inverse heat conduction problems. This method uses symbols to represent the temporal source strength and then executes a computational method to calculate the temperature distribution. Consequently, a set of linear equations is constructed from the comparison between the calculated symbolic temperature and the measured numerical temperature. Thus, the inverse problem is solved through the linear least-squares error method, which leads to a solution of the unknown source strength at the present time step. Results from the examples confirm that the proposed method is applicable in solving the inverse heat source problem.

INTRODUCTION

The inverse heat conduction problems deal with the determination of the crucial parameters in analysis such as the internal energy source, contact conductance, surface heat flux, thermal properties, etc. They have been widely applied in many design and manufacturing problems especially when the direct measurements for the problem are not possible. Wide attention has been called to this problem, and most studies employ the numerical methods [1–8] to determine the unknown conditions in the inverse problems. In the numerical methods, the inverse problem is formulated from the finite difference, the finite element, or the boundary element methods to calculate the responses of the system. In order to determine unknown conditions, these methods have often been combined with the optimization algorithms such as regularization technique, the sequential regularization approach, the adjoint equation approach coupled to the conjugate gradient method, genetic algorithms, and the multi-dimensional simplex method.

Based on the numerical approach, it needs iteratively solve the direct and the sensitivity problems in order to obtain the undetermined conditions. The direct problem is used to supply a solution that is compared to the measured responses based on the presumed values of the unknown condition. The sensitivity problem is used to offer the search direction and the search step size in the optimization algorithm. The sensitivity calculation not only increases the computational load but also limits the minimal step of the temporal coordinate. To avoid the above problems, Yang [9] successfully applied a symbolic method to solve the inverse heat conduction problem. The unique feature of the symbolic approach is that the inverse computation is in a linear domain and the nonlinear optimization process used in the numerical approach can be eliminated. Furthermore, the iteration in the direct problem is avoided and the calculation in the sensitivity analysis is eliminated. However, the side effects of the symbolic computation are also presented, which is the unpredictable growing size of the memory allocations and leads to an inefficient computation [10]. To resolve the problem, a sequential algorithm is proposed and the algorithm still has the advantage in the usage of the symbolic computation. Additionally, the concept of the future time [4] is also employed to stabilize the estimated results.

In the process of the proposed approach, the source strength is represented as a symbol in each time step. A hybrid method, a finite-element method in spatial domain and a finite-difference method in temporal domain [11], is performed to find the temperature field in the slab. In each time step, the calculated results of temperature distribution are expressed by an unknown symbol that describes the source strength at the present time step. Because the calculated temperature field is explicitly given as function of the source strength, the inverse analysis can be directly achieved by the resulting symbolic expressions and measurement datum. As a result, the inverse problem becomes a linear equation with one unknown variable in each time step. It leads to a solution of the unknown source strength through a direct substitution. When the concept of future time $(=r)$ is used, the inverse

NOMENCLATURE

a	coefficient in the derived temperature equation
$g(t)$	estimated source strength
r	number of future time
t	temporal coordinate
T, Y	temperature
x	spatial coordinate
[ ]	matrix notation
{ }	vector notation.

Greek symbols

Δx	increment of spatial coordinate
Δt	increment of temporal domain
λ	random number
σ	standard deviation of measurement errors
ϕ	estimated source strength at a temporal grid.

Subscripts

i, j, k, m	indices
exact	exact temperature
measurement	measured temperature
s	measured location.

Superscripts

j, p	indices of temporal coordinate
$\bar{}$	dimensional parameters
$\hat{}$	estimated strength.

problem becomes a problem with r linear equations. It leads to a solution of the unknown source strength through a linear least-squares error method. In this paper, only the linear case is considered. It means that there are no temperature-dependent coefficients in the heat equation or in the boundary conditions. The dimensionless heat equation is considered with the sensor location at one side of the boundaries (i.e., $x = 1$), which can be implemented to other situations with constant coefficients and variable's length of the spatial domain. In nonlinear problems, the present analysis can be used to compute the associated linearized equations. Furthermore, it is not difficult to extend our analysis to the case of multiple sensor locations.

This paper includes four sections. In the first section, the current development of the inverse problems is introduced and the features of the proposed method are also stated. In the second section, the characteristics of the inverse problem are delineated and the process of the proposed method is illustrated. In the third section, three different heat sources are employed to demonstrate the process of the proposed method. A discussion of the analyzed results is also presented in this section. At last, the overall contribution of this research is concluded in the fourth section.

APPROACH TO A HEAT SOURCE PROBLEM BASED ON THE PROPOSED METHOD

Problem statement

The inverse heat source problem in one spatial dimensional consists of finding the strength of the heat source at interior point of the spatial interval while the temperature measurements at the end are given. Consider a slab with $\bar{L}$ thickness and constant thermal properties. This slab originally has a uniformly distributed temperature. At a specific time $\bar{t} = 0$, a heat source $\bar{g}(t)$ is applied to the interior of the slab $\bar{x} = \bar{x}_s$ while the front and back surfaces are adiabatic. Then, a dimensionless mathematical formation of the heat conduction problem is presented as follows:

$$\frac{\partial^2 T}{\partial x^2} + g(t)\delta(x - x_s) = \frac{\partial T}{\partial t} \quad 0 < x < 1, \quad t > 0 \qquad (1)$$

$$T(x, 0) = 0 \quad 0 \leqslant x \leqslant 1 \qquad (2)$$

$$\left.\frac{\partial T}{\partial x}\right|_{x=0} = 0 \quad t > 0 \qquad (3)$$

$$\left.\frac{\partial T}{\partial x}\right|_{x=1} = 0 \quad t > 0 \qquad (4)$$

where the following dimensionless quantities are defined as:

$$x = \frac{\bar{x}}{\bar{L}} \quad x_s = \frac{\bar{x}_s}{\bar{L}} \quad T = \frac{\bar{T}}{\bar{T}_0} \quad k = \frac{\bar{k}}{\bar{k}_r} \quad t = \frac{\bar{k}_r}{\bar{\rho}\bar{C}_p}\frac{\bar{t}}{\bar{L}^2}$$

$\bar{k}$ is the thermal conductivity and $\bar{\rho}\bar{C}_p$ is the heat capacity per unit volume, $\bar{T}_0$ and $\bar{k}_r$ refer to the non-zero reference temperature and thermal conductivity, respectively. We assume $\bar{k} = \bar{k}_r$ and $g(t) = \bar{g}(t)\bar{L}^2/\bar{k}_r\bar{T}_0$.

The inverse problem is given the temperature $Y(t)$ measured at $x = 1$ to estimate the strength of the heat source $g(t)$.

The method to determine the strength of the heat source

The proposed method uses a finite-element method with a linear element to discretize the spatial coordinate and uses a finite-difference method to discretize the temporal coordinate. A finite-element method with p equidistant grid at $t = t_i$ [11] is used to construct the following matrix equation:

$$[N]\{\dot{T}_i\} = \{\Phi_i\} - [M]\{T_i\} \tag{5}$$

where $M =$

$$\begin{bmatrix} \frac{1}{\Delta x} & -\frac{1}{\Delta x} & \cdots & 0 & 0 \\ -\frac{1}{\Delta x} & \frac{2}{\Delta x} & \cdots & 0 & 0 \\ 0 & \cdots & \cdots & 0 & 0 \\ 0 & \cdots & -\frac{1}{\Delta x} & \frac{2}{\Delta x} & -\frac{1}{\Delta x} \\ 0 & \cdots & 0 & -\frac{1}{\Delta x} & \frac{1}{\Delta x} \end{bmatrix}_{(p+1)\times(p+1)} \tag{6}$$

$$[N] = \begin{bmatrix} \frac{\Delta x}{3} & \frac{\Delta x}{6} & \cdots & 0 & 0 \\ \frac{\Delta x}{6} & \frac{2\Delta x}{3} & \cdots & 0 & 0 \\ 0 & \cdots & \cdots & 0 & 0 \\ 0 & \cdots & \frac{\Delta x}{6} & \frac{2\Delta x}{3} & \frac{\Delta x}{6} \\ 0 & \cdots & 0 & \frac{\Delta x}{6} & \frac{\Delta x}{3} \end{bmatrix}_{(p+1)\times(p+1)} \tag{7}$$

$$\{\Phi_i\} = \begin{Bmatrix} 0 \\ \cdots \\ \phi_i \\ \cdots \\ 0 \end{Bmatrix}_{(p+1)} \qquad \{\dot{T}_i\} = \begin{Bmatrix} \dot{T}_i^1 \\ \dot{T}_i^2 \\ \cdots \\ \dot{T}_i^p \\ \dot{T}_i^{p+1} \end{Bmatrix}_{(p+1)}$$

$$\{T_i\} = \begin{Bmatrix} T_i^1 \\ T_i^2 \\ \cdots \\ T_i^p \\ T_i^{p+1} \end{Bmatrix}_{(p+1)} \tag{8}$$

where $\phi_i = g(t_i)$ and $\dot{T}_i^j = (\mathrm{d}T_i^j/\mathrm{d}t)$. Δx is the increment of the spatial coordinate. The superscript is denoted as the index of the spatial grid and the subscript is denoted as the index of the temporal grid. The location of ϕ_i in vector $\{\Phi_i\}$ is at the grid corresponding with the source location x_s.

Next, we consider our finite element expression for $\{\dot{T}_i\}$ as a backward difference at time t_i. Therefore, we have

$$\{\dot{T}_i\} = \frac{1}{\Delta t}\{T_i\} - \frac{1}{\Delta t}\{T_{i-1}\}. \tag{9}$$

Here Δt is the increment of the temporal coordinate. Substitute equation (9) into equation (5), we have the following differential equation:

$$[K]\{T_i\} = [B]\{T_{i-1}\} + \{\Phi_i\} \tag{10}$$

where

$$[K] = [M] + \frac{1}{\Delta t}[N] \quad \text{and} \quad [B] = \frac{1}{\Delta t}[N].$$

Therefore, the strength of the heat source ϕ_i can be solved iteratively along the temporal coordinate. An important characteristic of the method is its sequential nature; that is the estimated variable depends on the measured temperature and the previous ϕs and i is sequentially increased by one at each time step.

When $t = t_i$, the temperature distribution $\{T_i\}$ can be derived from equation (10) as follows:

$$\begin{aligned}\{T_i\} &= [K]^{-1}[B]\{T_{i-1}\} + [K]^{-1}\{\Phi_i\} \\ &= [C]\{T_{i-1}\} + [D]\{\Phi_i\} \\ &= [C]\{T_{i-1}\} + [D]\{u_i\}\phi_i \end{aligned} \tag{11}$$

where $[C] = [K]^{-1}[B]$ and $[D] = [K]^{-1}$. $\{u_i\}$ is a unit vector with a unit at i-component. Therefore, the calculated temperature at j spatial grid and at time t_i can be represented as a linear function of the estimated strength ϕ_i.

$$T_i^j = a_{i,0}^j + a_{i,i}^j\phi_i \tag{12}$$

where $a_{i,0}^j$ is calculated from the previous state and $a_{i,i}^j$ is a numerical value corresponding with the coordinate at t_i and x_j. In other words, the value of $a_{i,0}^j$ is the i-component of vector $[C]\{T_{i-1}\}$ and the value of $a_{i,i}^j$ is the i-component of vector $[D]\{u_i\}$.

When $t = t_m$, the estimated conditions $\hat{\phi}_1$, $\hat{\phi}_2$, $\hat{\phi}_3, \ldots, \hat{\phi}_{m-1}$ are known and the object is to estimate $\hat{\phi}_m$. In order to add stability to the inverse algorithm, the simplest sequential procedure is to assume temporally that several future conditions are constant with time. In other words, the undetermined conditions $\phi_m, \phi_{m+1}, \ldots, \phi_{m+r-2}, \phi_{m+r-1}$ are assumed to be equal:

$$\phi_{m+1} = \cdots = \phi_{m+r-2} = \phi_{m+r-1} = \phi_m. \tag{13}$$

The calculate temperature at j-spatial grid and at time $t_m, t_{m+1}, \ldots, t_{m+r-1}$ can be written as follows:

$$\begin{aligned} T_m^j &= a_{m,0}^j + a_{m,m}^j\phi_m \\ T_{m+1}^j &= a_{m+1,0}^j + a_{m+1,m}^j\phi_m + a_{m+1,m+1}^j\phi_{m+1} \\ &= a_{m+1,0}^j + (a_{m+1,m}^j + a_{m+1,m+1}^j)\phi_m \\ T_{m+2}^j &= a_{m+2,0}^j + a_{m+2,m}^j\phi_m + a_{m+2,m+1}^j + a_{m+2,m+2}^j\phi_{m+2} \\ &= a_{m+2,0}^j + (a_{m+2,m}^j + a_{m+2,m+1}^j + a_{m+2,m+2}^j)\phi_m \\ &\cdots \end{aligned}$$

$$
\begin{aligned}
T^j_{m+r-1} &= a^j_{m+r-1,0} + a^j_{m+r-1,m}\phi_m + a^j_{m+r-1,m+1}\phi_{m+1} \\
&\quad + a^j_{m+r-1,m+2}\phi_{m+2} + \cdots + a^j_{m+r-1,m+r-2}\phi_{m+r-2} \\
&\quad + a^j_{m+r-1,m+r-1}\phi_{m+r-1} \\
&= a^j_{m+r-1,0} + (a^j_{m+r-1,m} + a^j_{m+r-1,m+1} + a^j_{m+r-1,m+2} \\
&\quad + \cdots + a^j_{m+r-1,m+r-2} + a^j_{m+r-1,m+r-1})\phi_m.
\end{aligned} \tag{14}
$$

A linear least-squares error method [12] is used to estimate the value of ϕ_m. Therefore, the result $\hat{\phi}_m$ is

$$
\hat{\phi}_m = \frac{\sum_{i=m}^{m+r-1} \sum_{k=m}^{i} a^j_{i,k} \times (Y^j_i - a^j_{i,0})}{\sum_{i=m}^{m+r-1} \left(\sum_{k=m}^{i} a^j_{i,k} \right)^2} \tag{15}
$$

where Y^j_i is the measurement temperature at $t = t_i$ and $x = x_j$. This equation provides an algorithm that is used in a sequential manner by increasing temporal index by one for each time step.

The above formulation is derived from a finite-element-difference approach to estimate the strength of the heat source at $x = x_s$ when the temperature measurements are taken from $x = x_j \neq x_s$. It is not difficult to extend the proposed method to estimate multiple sources, to adopt multi-sensor's measurement, or to adopt the different kinds of numerical methods in the inverse estimation.

COMPUTATIONAL ALGORITHM

The procedure of the proposed method can be summarized as follows: given the number of future time r and the discretized spatial and temporal size Δx and Δt. The problem is to estimate $\hat{\phi}_m$ when $t = t_m$. The previous states $\hat{\phi}_1, \hat{\phi}_2, \hat{\phi}_3, \ldots, \hat{\phi}_{m-1}$ are known and the calculated temperature distribution at $t = t_{m-1}$ over slab is available. The measured temperatures are taken from $x = x_j$.

Step 1. Construct equation (14) through the numerical method.
Step 2. Collect the coefficients of equation (14).
Step 3. Collect the measurement $Y^j_m, Y^j_{m+1}, \ldots, Y^j_{m+r-1}$.
Step 4. Calculate $\hat{\phi}_m$ according to equation (15).
Step 5. Calculate the temperature distribution at $t = t_m$.
Step 6. Terminate the process if the final iteration is attached. Otherwise, let $m = m+1$ and return to step 1.

RESULTS AND DISCUSSIONS

In this section, problems defined from equations (1)–(4) are used as examples to estimate the strength of the heat source. Three different source functions over temporal domain; namely, a triangular function, a sinusoidal function, and a quarter sinusoidal function are used to illustrate the proposed method. In the example problems, the stability and the accuracy of the estimation are discussed. Furthermore, the results are also compared to the solutions of Huang and Ozisik's approach [13]. The exact temperature and the source strength used in the following examples are selected so that these functions can satisfy equations (1)–(4). The accuracy of the proposed method is assessed by comparing the estimated source strength with the preselected source strength. Meanwhile, the simulated temperature measurement is generated from the exact temperature in each problem and it is presumed to have measurement errors. In other words, the random errors of measurement are added to the exact temperature. It can be shown in the following equation:

$$
Y_{\text{measurement}} = Y_{\text{exact}} + \lambda\sigma \tag{16}
$$

where Y_{exact} in equation (16) is the exact temperature and $Y_{\text{measurement}}$ is the measured temperature at the grid points. σ is the standard deviation of measurement errors. λ is a uniform random number.

The time domain in all cases is from 0–1.8 with 0.03 increment. As well, the increment of spatial coordinate is 0.1. The heat sources are applied at $x = 0.5$ and the temperature measurements are taken from $x = 1$ in all examples. Two cases of random noise level $\sigma = 0.001$ and $\sigma = 0.005$ are adopted. The value λ is calculated by the IMSL subroutine DRNNOR [14] and chosen over the range $-2.576 < \lambda < 2.576$, which represent the 99% confidence bound for the measurement temperature.

The following three examples demonstrate the application of the proposed approach. The strength of the heat source is presented as the time-varying function. Detailed descriptions for the examples are shown as follows:

Example 1:

$$
g(t) = 0.3 + \tfrac{7}{9}t \quad \text{when} \quad 0 < t < 0.9
$$

and

$$
g(t) = 1.5 - \tfrac{5}{9}t \quad \text{when} \quad 0.9 \leqslant t \leqslant 1.8.
$$

Example 2:

$$
g(t) = \sin\left(\frac{5\pi t}{6}\right) \quad \text{when} \quad 0 < t \leqslant 1.8.
$$

Example 3:

$$
g(t) = 1.0 \quad \text{when} \quad 0 < t < 0.9
$$

and

$$
g(t) = 0.5 \quad \text{when} \quad 0.9 \leqslant t \leqslant 1.8.
$$

When measurement errors are not included, no future time ($r = 1$) is needed in the example problems. The results are shown in Figs. 1, 4, and 7, and all

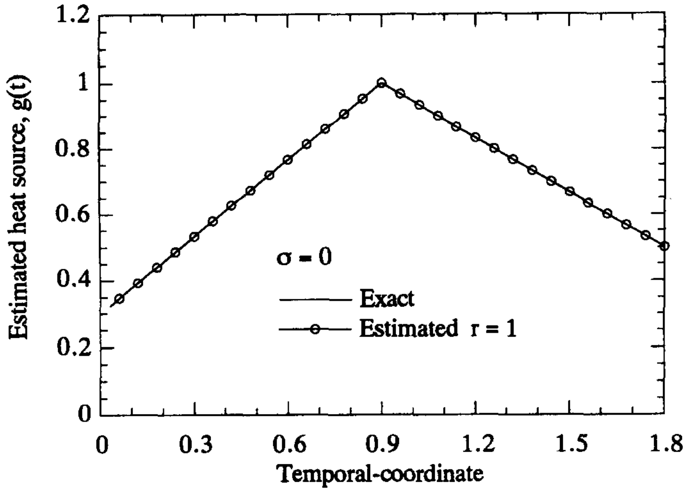

Fig. 1. Estimation of the strength of the heat source in example 1 (measurement error $\sigma = 0$).

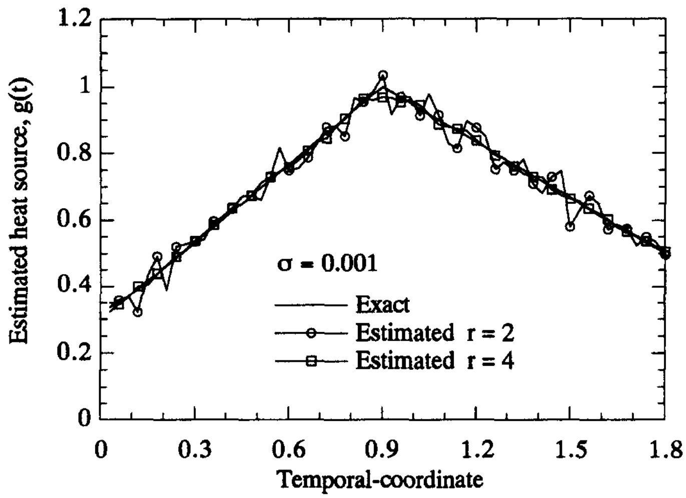

Fig. 2. Estimation of the strength of the heat source in example 1 (measurement error $\sigma = 0.001$).

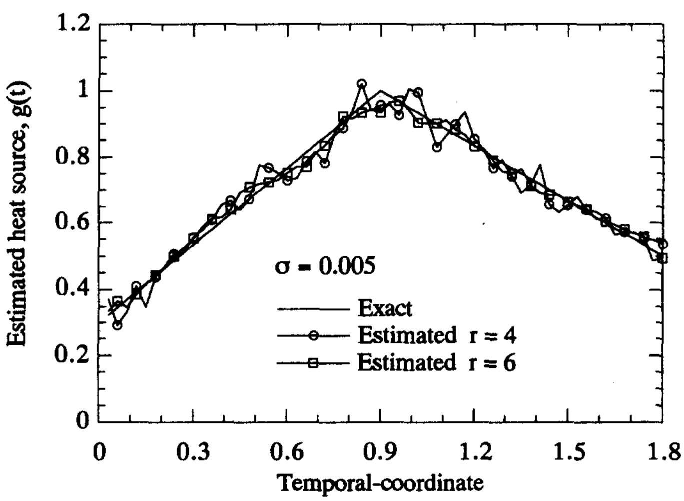

Fig. 3. Estimation of the strength of the heat source in example 1 (measurement error $\sigma = 0.005$).

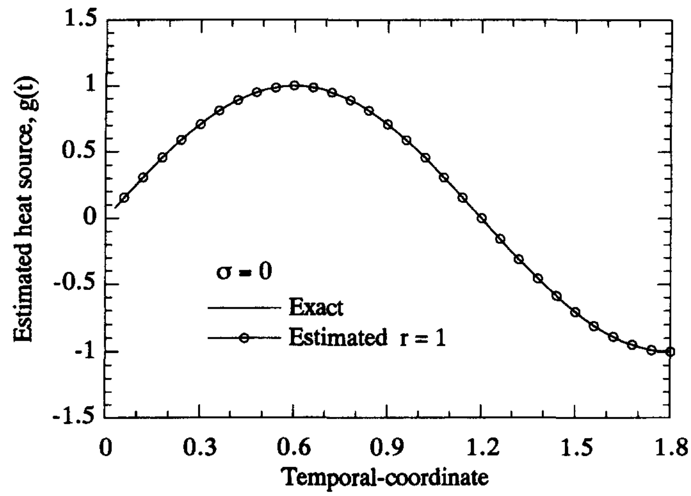

Fig. 4. Estimation of the strength of the heat source in example 2 (measurement error $\sigma = 0$).

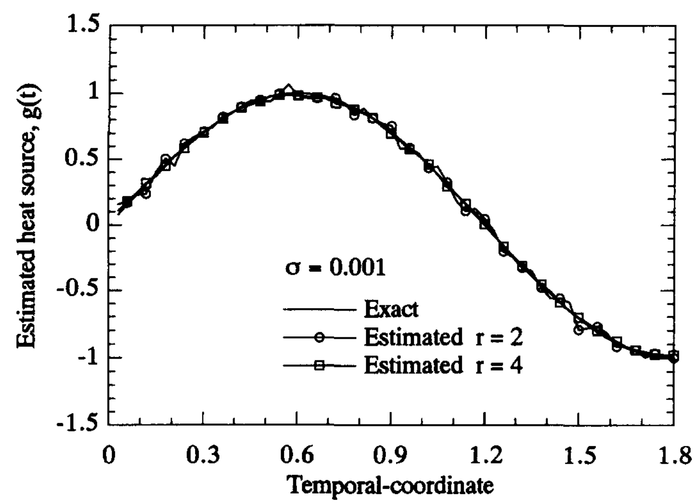

Fig. 5. Estimation of the strength of the heat source in example 2 (measurement error $\sigma = 0.001$).

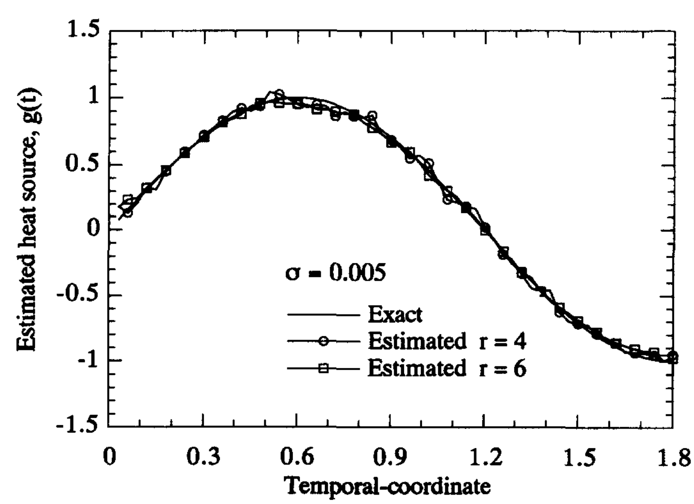

Fig. 6. Estimation of the strength of the heat source in example 2 (measurement error $\sigma = 0.005$).

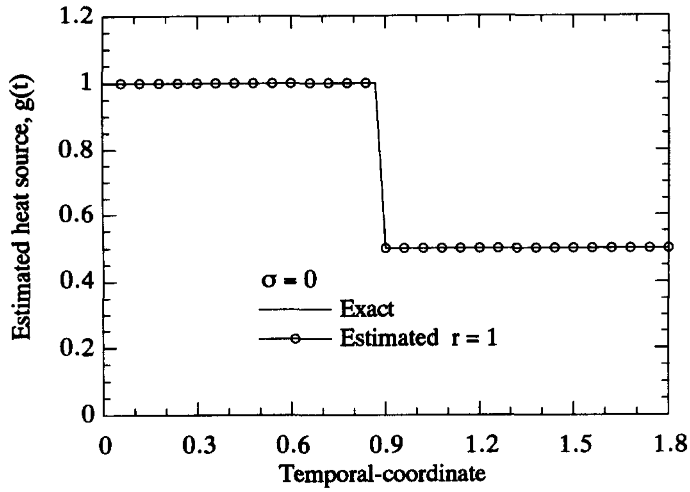

Fig. 7. Estimation of the strength of the heat source in example 3 (measurement error $\sigma = 0$).

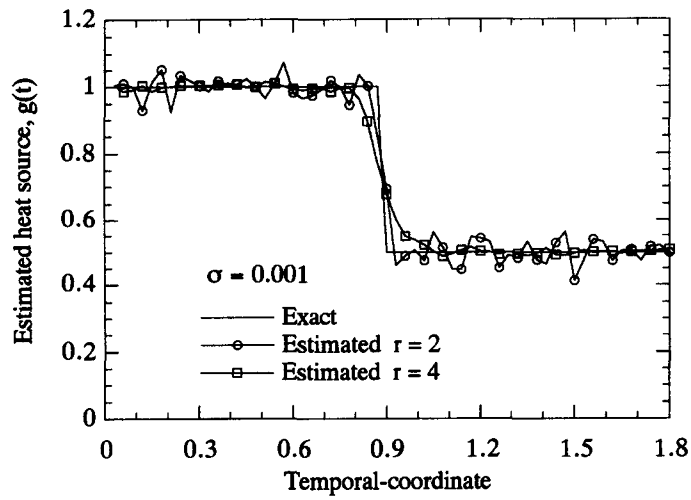

Fig. 8. Estimation of the strength of the heat source in example 3 (measurement error $\sigma = 0.001$).

examples have excellent approximations. When measurement errors are included, the future times are used to stabilize the estimated results in the example problems. In the example problems, when the future times are used, the measured value after the last time step $t = 1.8$ are generated from the same source function form as that in the previous time interval. For example, the measured value after the last time step $t = 1.8$ in example 1 are calculated from $g(t) = 1.5-(5/9)t$. In the first case $\sigma = 0.001$, the results are shown in Fig. 2 for example 1, Fig. 5 for example 2, and Fig. 8 for example 3. In the figures, the results show that they have more stable outcome when $r = 4$. In the second case $\sigma = 0.005$, the results are shown in Fig. 3 for example 1, Fig. 6 for example 2, and Fig. 9 for example 3. The results show that they have more stable estimation when $r = 6$. Generally speaking, the results show that it needs more future times to stabilize the estimation when error level is raised. The same examples are done by Huang and Ozisik [12]. They used the combination of the regular and modified conjugate gradient methods to perform inverse analysis. They concluded that the combined method is found to be more accurate than either of the regular and modified conjugate gradient methods. However, the results shown in their paper have significant deviation near the final time $t = 1.8$. To alleviate the error occurring in the immediate vicinity of the end point, they choose the solution valid up to a time less than $t = 1.5$. In other words, the agreement between the predicted and the exact values are good only over the time interval $0 < t < 1.5$. In this research, the above incorrect phenomenon at the ending of the time interval can be alleviated. The agreement between the estimated results and the exact ones are very good over the problem domain $0 < t \leqslant 1.8$.

In conclusion, the proposed method has more accurate results than those of the past approach. More-

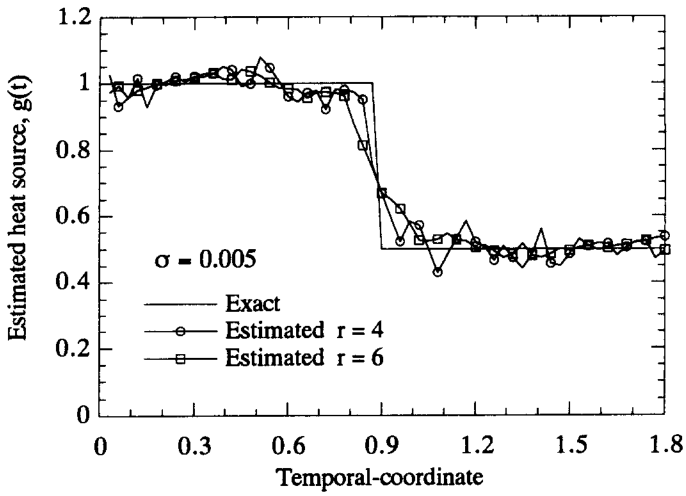

Fig. 9. Estimation of the strength of the heat source in example 3 (measurement error $\sigma = 0.005$).

over, the numerical results show that the exact solution can be found when future time is not used ($r = 1$). Yet, this is under the condition that the measurement errors are neglected. When measurement errors are included, it is suggested that the $r = 4$ for $\sigma = 0.001$ and $r = 6$ for $\sigma = 0.005$ are needed for the better estimations in the examples.

CONCLUSIONS

A sequential approach has been introduced for solving the inverse heat source problems. A finite-element-difference method is employed to solve the symbolic temperature field when the source strength is represented as a symbol. The proposed method has the advantage of the usage of the symbolic computation in the inverse problem but it eliminates the disadvantages caused by the symbolic computation. In other words, the unique feature of the proposed approach is that the inverse computation is in a linear domain, the nonlinear optimization process used in the numerical approach can be eliminated, the iteration in the direct problem is avoided, and the calculation in the sensitivity analysis is eliminated. Furthermore, the size of the computer memories can be reduced and it leads to an increase in efficiency in the symbolic computation. Three examples have been used to show the usage of the proposed method. The results show that the exact solution can be found through the proposed method when measurement errors are not considered. When measurement errors are increased, it is suggested that the increasing of the future times to stabilize the fluctuation of the estimation from the exact solution (i.e., $r = 4$ for $\sigma = 0.001$ and $r = 6$ for $\sigma = 0.005$). This result can be referred to adopt the number of future times in the inverse source problem in the future researches. The proposed method is also applicable to the other kinds of inverse problems such as boundary estimation in the one- or multi-dimensional heat transfer problems.

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